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COMMENT

Liftshitz-point critical behaviour to O(ϵ^2)

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Abstract

We comment on a recent letter by de Albuquerque and Leite (2001 *J. Phys. A: Math. Gen.* **34** L327), in which results to the second order in $\epsilon = 4 - d + \frac{m}{2}$ were presented for the critical exponents ν_{L2}, η_{L2} and γ_{L2} of d-dimensional systems at m-axial Lifshitz points. We point out that their results are at variance with ours. The discrepancy is due to their incorrect computation of momentum-space integrals. Their speculation that the field-theoretic renormalization group approach, if performed in position space, might give results different from when it is performed in momentum space is refuted.

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In a recent letter [1] de Albuquerque and Leite (AL) presented results to the second order in $\epsilon = 4 - d + \frac{m}{2}$ for the critical exponents ν_{L2}, η_{L2} and γ_{L2} of d-dimensional systems at m-axial Lifshitz points. For the special case $m = 1$ of a uniaxial Lifshitz point, these results were previously given in a (so far apparently unpublished) preprint [2]. The ϵ^2 terms AL found are at variance with ours [3, 4].

As an explanation for these discrepancies AL suggest the following. Both we as well as AL employed a field-theoretic renormalization group approach based on dimensional regularization. To compute the residues of the ultraviolet (UV) poles at $\epsilon = 0$, we found it convenient to perform (part of) the calculation in position space. By contrast, AL worked entirely in momentum space. They speculate [1] 'that calculations performed in momentum space and coordinate space are inequivalent, as far as the Lifshitz critical behaviour is concerned'.

This speculation is untenable and a serious misconception, a fact which should be obvious not only to readers with a background in field theory. The reason simply is: at each step of the calculation one can transform from momentum space to position space and *vice versa*.

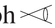
To become more specific, consider an N-point vertex function $\Gamma^{(N)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ of a dimensionally regularized, translationally invariant, renormalizable Euclidean field theory, such as the $|\phi|^4$ theory for an m-axial Lifshitz point considered both by us [3, 4] and AL. In momentum space the vertex functions have the form

$$\tilde{\Gamma}^{(N)}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)\,(2\pi)^d\,\delta\left(\sum_{i=1}^{N}\boldsymbol{q}_i\right)$$

where $\tilde{\Gamma}^{(N)}$ are conventional *functions* of $N-1$ independent momenta, e.g., $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{N-1}$. $\tilde{\Gamma}^{(N)}$ also depend on ϵ (i.e. on d): they are *meromorphic* in ϵ, having UV poles at $\epsilon = 0$. Since the issue is the ϵ expansion, these are the only poles we have to consider; possible other UV poles at special values $\epsilon > 0$ need not concern us here. Likewise, we do not have to worry about possible infrared poles one encounters in perturbation expansions about the Lifshitz point for a fixed space dimension d, nor embark on a discussion of related subtle questions such as the appearance of perturbatively non-accessible mass shifts meromorphic in ϵ and on how these problems are avoided in massive, fixed-d renormalization schemes.

The Fourier back-transforms of the *functions* $\tilde{\Gamma}^{(N)}$ define *generalized functions (distributions)*, which depend on the $N-1$ difference variables $\boldsymbol{x}_{j1} \equiv \boldsymbol{x}_j - \boldsymbol{x}_1$, $1 \leqslant j \leqslant N-1$. The same applies to each individual Feynman integral contributing to $\Gamma^{(N)}$.

Field-theoretical renormalization group approaches such as the ones on which AL's work and ours are based hinge on the possibility of introducing a well-defined *renormalized* theory by absorbing the UV singularities of the theory's primitively divergent vertex functions in a consistent manner through counter-terms that are *local in position space*. In order that the renormalization procedure can be interpreted as a re-parametrization, these counter-terms must have the form of the (local) interactions appearing in the original Hamiltonian, except for a finite number of admissible additive ones. Well-known mathematical renormalization theorems [5, 6] ensure that this is the case, order by order in perturbation theory.

Central to the proofs of such renormalization theorems is the observation that the primitive UV singularities have a *local* structure in *position space*. It is precisely this property that is crucial for the renormalizability of the theory. It ensures that the counter-terms, computed to a given order of perturbation theory, provide the subtractions for all divergent subintegrations of the Feynman graphs of the next higher order that are required to cancel *all those UV singularities that do not have the form of local counter-terms.* Such non-local UV singularities occur indeed: for instance, the graph  has momentum-dependent pole terms $\sim\epsilon^{-1}$ (involving logarithms of momenta). These are due to the divergent subintegral ; they do *not* have the form of local counter-terms but cancel upon making the appropriate subtraction for this subgraph. (This subtraction is produced by part of the one-loop counter-term $\propto \phi^4$, see, e.g., section 3.B of [7].) Zimmermann's forest formula [8] clarifies precisely which subtractions have to be made for each individual Feynman graph. The locality of the counter-terms manifests itself in the fact that in the final subtractions which must be made for superficially divergent graphs the graph is *shrunk to a point*.

What we have just explained has been known for decades and can be found in standard textbooks on field theory. It is true that for computational reasons many authors prefer the momentum representation when explaining the renormalization procedure. Therefore, the significance of the UV singularities' local structure in position space may escape the reader's attention if not properly emphasized. However, a very clear exposition of the importance of this locality has already been given in one of the earliest classic textbooks on renormalization [9].

The renormalization procedure can be performed equally well in momentum or position space. Utilizing dimensional regularization in conjunction with minimal subtraction of poles is advantageous in that one does not have to worry about how the regularization scheme and the conditions for fixing the counter-terms translate upon Fourier transformation: the scheme can be applied equally well in the momentum or position representation. The upshot of these considerations is that *there is no way that AL's and our calculation can both be correct.*

The source of the discrepancies between AL's work and ours can be traced back to the different results they find for the required two-loop integrals. For example, our result for the integral $I_3(p,k)$ defined in equation (3) of AL's paper [1] reads

$$I_3(p,k) = \frac{(2\pi)^{2d^*}}{\epsilon}\left[\frac{j_\sigma(m)\,k^4}{16\,m\,(m+2)} - \frac{j_\phi(m)\,p^2}{2\,(8-m)}\right] + \mathrm{O}(\epsilon^0) \tag{1}$$

with

$$j_\phi(m) = \frac{2^{10+m}\,\pi^{6+\frac{3m}{4}}\,\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(2-\frac{m}{4}\right)\,\Gamma^2\left(\frac{m}{4}\right)}\int_0^\infty \mathrm{d}\upsilon\,\upsilon^{m-1}\,\Phi^3(\upsilon; m, d^*) \tag{2}$$

where

$$\Phi(\upsilon; m, d^*) = \int\frac{\mathrm{d}^{d^*-m}p}{(2\pi)^{d^*-m}}\int\frac{\mathrm{d}^m k}{(2\pi)^m}\frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{e}+\boldsymbol{k}\cdot\boldsymbol{\upsilon})}}{p^2+k^4} \tag{3}$$

is the scaling function associated with the free critical propagator in position space (cf. equation (13) of [3]), at the upper critical dimension $d^* = 4+\frac{m}{2}$. Here $\boldsymbol{e}$ is a unit d^*-m vector, while $\boldsymbol{\upsilon}$ is an arbitrarily directed m-vector. The integral $j_\sigma(m)$ is similar to $j_\phi(m)$, except that its integrand has an additional factor υ^4.

From AL's equations (11) and (18), we can infer their result for $I_3(p,0)$; it reads

$$I_3^{(\mathrm{AL})}(p,0) = -\pi^{4+\frac{m}{2}}\,\frac{\Gamma^2\left(\frac{m}{4}\right)}{\Gamma^2\left(\frac{m}{2}\right)}\,\frac{1}{8-m}\,\frac{p^2}{\epsilon} + \mathrm{O}(\epsilon^0). \tag{4}$$

To see that this cannot be correct, one must merely consider the isotropic case $m=d=8-\epsilon$: for this, AL's result (4) predicts a pole $\propto \epsilon^{-2}$, even though the pole part $\propto p^2$ *must vanish* because $\boldsymbol{p}$ has $d-m=0$ components. By contrast, our result (1)–(3) does not violate this condition since $j_\phi(8)=0$. (See sections 4.5 and 4.4 of [4] where we verified that our ϵ-expansion results for general values of m reduce to known ones in both isotropic cases $m=d$ and $m=0$, respectively.)

AL realized the incorrectness of their findings for $m=8$. Yet they seem to believe that the 'approximations' they made in their computation of $\ell \geqslant 2$ loop integrals do not lead to erroneous results. Details of their approximations are described in [2]. The crux of their method is 'to impose the constraint' $\boldsymbol{k}_1 = -2\boldsymbol{k}_2$ on the momenta of the internal integral

$$I_2(\boldsymbol{p}_1+\boldsymbol{p},\boldsymbol{k}_1) = \int\frac{\mathrm{d}^{d-m}p_2\,\mathrm{d}^m k_2}{\left(p_2^2+k_2^4\right)\left[(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p})^2+(\boldsymbol{k}_1+\boldsymbol{k}_2)^4\right]} \tag{5}$$

of

$$I_3(p,0) = \int\frac{\mathrm{d}^{d-m}p_1\,\mathrm{d}^m k_1}{p_1^2+k_1^4}\,I_2(\boldsymbol{p}_1+\boldsymbol{p},\boldsymbol{k}_1). \tag{6}$$

This amounts to modifying the momentum term $(\boldsymbol{k}_1+\boldsymbol{k}_2)^4$ of the last propagator in equation (5) to k_2^4. The error this introduces is given by the analogue of the integral (6) one obtains through replacement of $I_2(\boldsymbol{p}_1+\boldsymbol{p},\boldsymbol{k}_1)$ by the corresponding difference $\delta I_2(\cdot,k_1) \equiv I_2(\cdot,\boldsymbol{k}_1) - I_2(\cdot,0)$, namely

$$\delta I_2(\boldsymbol{p}_1+\boldsymbol{p},k_1) = \int\frac{\mathrm{d}^{d-m}p_2\,\mathrm{d}^m k_2}{\left(p_2^2+k_2^4\right)\left[(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p})^2+(\boldsymbol{k}_1+\boldsymbol{k}_2)^4\right]} \times\frac{k_1^4+4\left(k_1^2+k_2^2\right)\,\boldsymbol{k}_1\cdot\boldsymbol{k}_2+6\,k_1^2\,k_2^2}{\left[(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p})^2+k_2^4\right]}. \tag{7}$$

Now the pole term $\propto p^2/\epsilon$ of I_3 we are concerned with corresponds to a logarithmic UV divergence $\sim p^2\ln\Lambda$ at the upper critical dimension ($\Lambda =$ cut-off). In order for AL's approximation to be correct, δI_2 must have no contributions that vary as $p^2\,p_1^{-2}$ or $p^2\,k_1^{-4}$

as $p_1 \sim k_1^2 \sim \Lambda \to \infty$. As can be seen for instance by power counting, this condition is *not satisfied*. (Readers preferring more mathematical scrutiny might want to compute $\nabla_{\boldsymbol{p}}^2\, \delta I_2$ and study its behaviour for large p_1 and k_1.) Accordingly, AL's approximation is unjustified whenever $m \neq 0$. The same kind of approximations are employed by AL for other $\ell \geqslant 2$ loop integrals.

In conclusion, let us outline how the pole term $\propto p^2/\epsilon$ of I_3 given in equation (6) can be recovered via a momentum-space calculation. Using a Schwinger representation for each one of the three propagators in equations (5) and (6) and performing the Gaussian integrations over $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, we obtain

$$I_3(p,0) = \pi^{d-m} \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}z\, (xy+yz+zx)^{-\frac{d-m}{2}} \times \int \mathrm{d}^m k_1 \int \mathrm{d}^m k_2 \exp\left(-\frac{xyz\, p^2}{xy+yz+zx} - x\, k_1^4 - y\, k_2^4 - z\, |\boldsymbol{k}_1 + \boldsymbol{k}_2|^4 \right). \tag{8}$$

Next, we make the variable transformations $X = x/z$, $Y = y/z$ and $\boldsymbol{K}_{1,2} = z^{1/4}\boldsymbol{k}_{1,2}$, and take the derivative $-\partial/\partial p^2\big|_{p=1}$ inside the integrals. The integration over z can now be performed; it produces the factor $\Gamma(\epsilon)(1+X^{-1}+Y^{-1})^{\epsilon} = 1/\epsilon + \mathrm{O}(\epsilon^0)$. Upon transforming to the variables $s = 1/X$ and $t = 1/Y$, one finds that

$$\frac{-\partial I_3(p,0)}{\partial p^2} = \frac{\pi^{d^*-m}}{\epsilon} \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t\, (st)^{-\frac{m}{4}}\, (1+s+t)^{\frac{m}{4}-3} \times \int \mathrm{d}^m K_1 \int \mathrm{d}^m K_2\ \exp\left(-\frac{K_1^4}{s} - \frac{K_2^4}{t} - |\boldsymbol{K}_1 + \boldsymbol{K}_2|^4 \right) + \mathrm{O}(\epsilon^0). \tag{9}$$

This is in conformity with equations (1)–(3). To see this, note that the integral j_ϕ is proportional to $\int \mathrm{d}^m \upsilon\, \Phi^3$. In momentum space, this is a convolution of the form $\int_{\boldsymbol{k}_1,\boldsymbol{k}_2} \tilde{\Phi}_{\boldsymbol{k}_1} \tilde{\Phi}_{\boldsymbol{k}_2} \tilde{\Phi}_{-\boldsymbol{k}_1-\boldsymbol{k}_2}$. The Fourier transform $\tilde{\Phi}_{\boldsymbol{k}}$ can be read from equation (14) of [3]; it involves a modified Bessel function $K_\nu(k^2)$, for which we use the representation

$$k^{2\nu}\, K_\nu(k^2) = 2^{\nu-1} \int_0^\infty \mathrm{d}x\, x^{\nu-1}\, \mathrm{e}^{-x-\frac{k^4}{4x}} \tag{10}$$

with integration variables x, y and z. Employing the transformations $s = x/z$, $t = y/z$ and $\boldsymbol{K}_{1,2} = z^{-1/4}\boldsymbol{k}_{1,2}$, we perform the integration over z. The result is the residue of the pole (9).

To summarize: AL's results are incorrect because of their unacceptable approximations made in computing the Feynman diagrams. Their speculation that the field-theoretic RG approach might yield different results depending on whether it is performed in position or momentum space does not hold.

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